A symbolic calculus on Symmetric Net arc functions: applications

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Stochastic Symmetric Nets

- A Colored Petri Net-like formalism proposed in the ’90s
- A «color syntax» devised to exploit (most) model symmetries during analysis
  - State space methods can benefit: lumpabiliy is exploited, (much) smaller state spaces are generated directly
  - Structural properties derivation (interesting features of PNs): define a language of expressions similar to arc functions and operators on them
  - The language for expressing structural properties is useful also for the generation of a set of *Symbolic* Ordinary Differential Equations to derive expected values of performance indices of interest.

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Deriving Symbolic ODE from SSNs without unfolding

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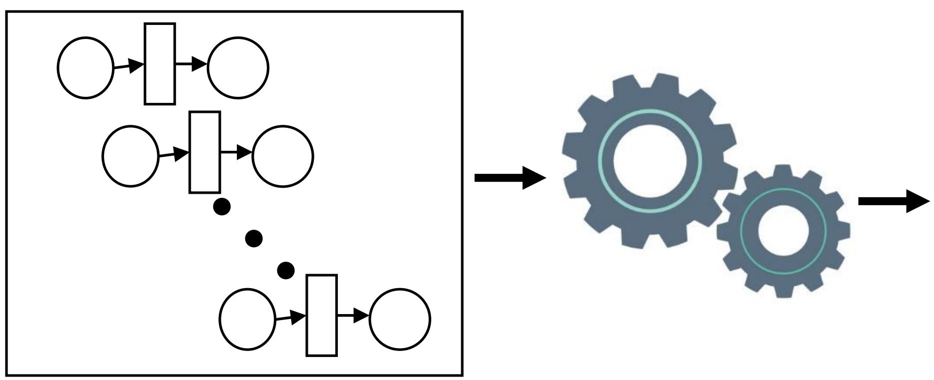
Fluidification of Stochastic Petri Nets

Analysis methods that approximate the behavior of the stochastic process underlying an SPN with a deterministic one, modelled through a system of Ordinary Differential Eqs. (under given assumptions [2][9]). Expected value at time T of state components can be obtained by solving the system of ODE.



$dx_1/dt = +\lambda x_2 - \mu\, x_3$

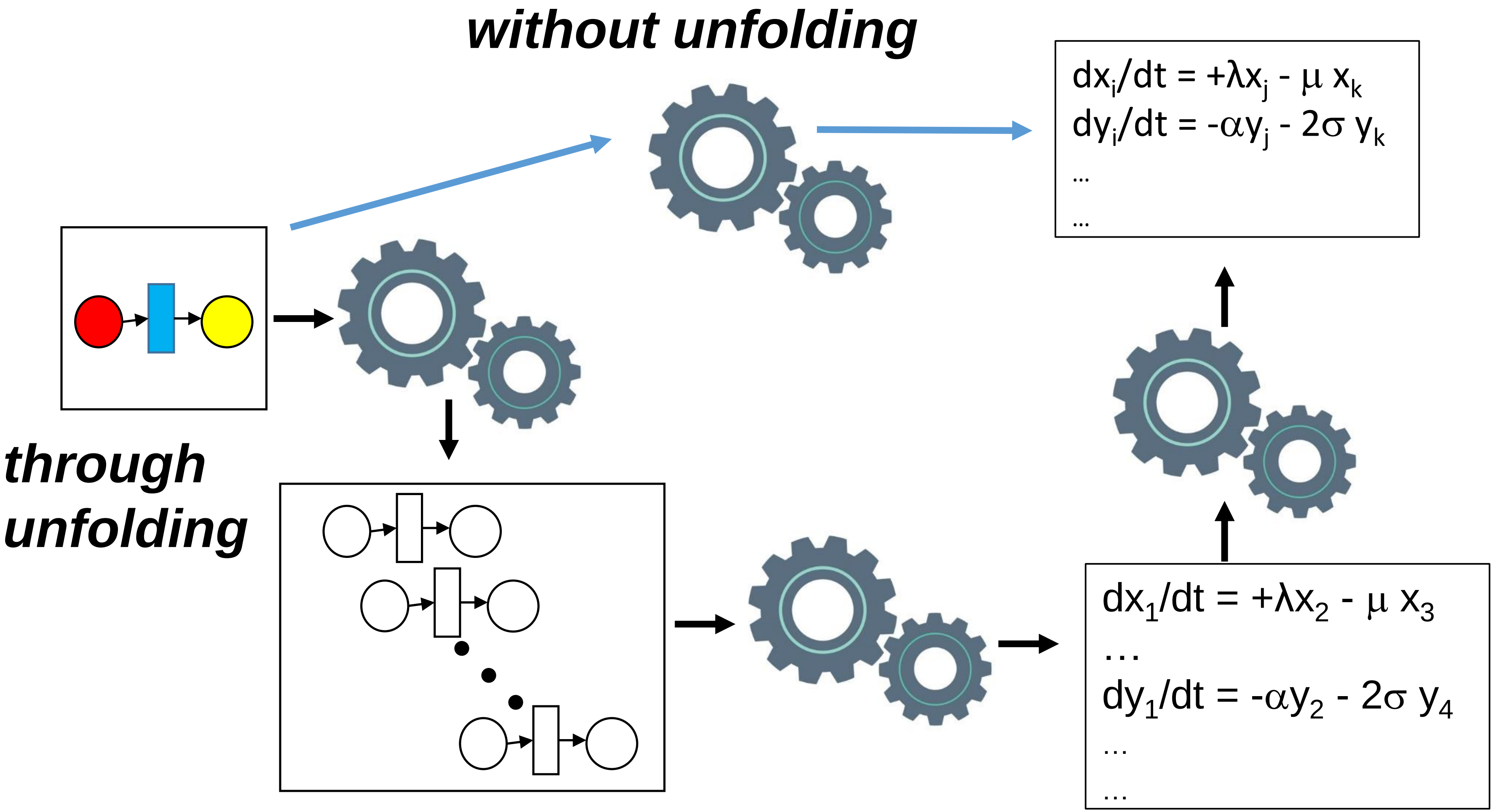
…

$dy_1/dt = -\alpha y_2 - 2\sigma\, y_4$

…

…

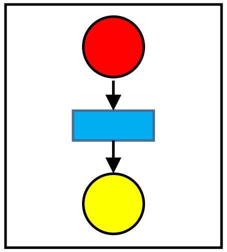
Fluidification of Stocastic Symmetric Nets



Outline

- Stochastic Symmetric Nets
- Ordinary Differential Equations vs Symbolic ODE
- Direct generation of SODE:
  - SSN partial unfolding
  - Symbolic ODE terms: transposing arc functions
  - Evaluation of the enabling degree of symbolic instances
  - Cardinality of symbolic expressions
- Tool support
- Concluding remarks
- Related work and Future work

Stochastic Symmetric Nets



Petri nets with colored tokens and transition instances + stochastic (exponentially distributed) firing times

- Compact representation of models
- Structured syntax of colors
  - Lumped State Space and Stochastic Process (CTMC)
  - Symbolic computation of structural properties
- Unfolding of an SSN model into a SPN enables the application of analysis methods defined only for SPNs (e.g. fluid approximation).

Color domain

$Mac \times Loc$

Arc functions: I(t,p), O(t,p)

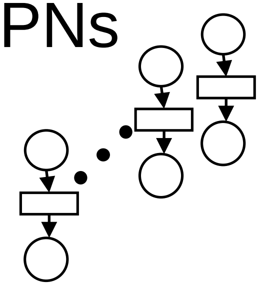
$\langle x, l\rangle[x \in N]$

$\langle l, l'\rangle[l \neq l']$

Expressions (arc fn. –like) and Operators

$[c_3 \neq c_2]\langle S - c \cap S_{C1}, c, c'\rangle[c \in C1]$

$(f_1 - f_2)^t \bullet f_3$



Exploiting symmetries

Groups of places in the unfolded model have «similar» ODE $\Rightarrow$ under the hypothesis of SYMMETRIC INITIAL MARKING the expected number of tokens in these places remains uniform in time [4].

Only one representative in each «equivalence class» is used to derive the solution. A Symbolic ODE is an ODE representing the whole class.

[4] M. Beccuti, C. Fornari, G. Franceschinis, S. Halawani, O. Ba-Rukab, A. Ahmad, and G. Balbo. From symmetric nets to differential equations exploiting model symmetries. Computer Journal, 58(1):23-39, 2015.

Exploiting symmetries

$$\begin{aligned}
\frac{dx_{P_{1,\alpha}}}{dv} &= -\lambda(\min[x_{P_{1,\alpha}}, x_{P_{2,\beta}}]) - \lambda(\min[x_{P_{1,\alpha}}, x_{P_{2,\delta}}]) \\
\frac{dx_{P_{1,\beta}}}{dv} &= -\lambda(\min[x_{P_{1,\beta}}, x_{P_{2,\alpha}}]) - \lambda(\min[x_{P_{1,\beta}}, x_{P_{2,\delta}}]) \\
\frac{dx_{P_{1,\delta}}}{dv} &= -\lambda(\min[x_{P_{1,\delta}}, x_{P_{2,\alpha}}]) - \lambda(\min[x_{P_{1,\delta}}, x_{P_{2,\beta}}])
\end{aligned}$$

$$\begin{aligned}
\frac{dx_{P_{2,\alpha}}}{dv} &= -\lambda(\min[x_{P_{1,\beta}}, x_{P_{2,\alpha}}]) - \lambda(\min[x_{P_{1,\delta}}, x_{P_{2,\alpha}}]) \\
\frac{dx_{P_{2,\beta}}}{dv} &= -\lambda(\min[x_{P_{1,\alpha}}, x_{P_{2,\beta}}]) - \lambda(\min[x_{P_{1,\delta}}, x_{P_{2,\beta}}]) \\
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\end{aligned}$$

$$\begin{aligned}
\frac{dx_{P_{3,\alpha,\beta}}}{dv} &= +\lambda(\min[x_{P_{1,\alpha}}, x_{P_{2,\beta}}]) \\
\frac{dx_{P_{3,\alpha,\delta}}}{dv} &= +\lambda(\min[x_{P_{1,\alpha}}, x_{P_{2,\delta}}]) \\
\frac{dx_{P_{3,\beta,\alpha}}}{dv} &= +\lambda(\min[x_{P_{1,\beta}}, x_{P_{2,\alpha}}]) \\
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\frac{dx_{P_{3,\delta,\beta}}}{dv} &= +\lambda(\min[x_{P_{1,\delta}}, x_{P_{2,\beta}}])
\end{aligned}$$

$$\begin{aligned}
\frac{d\widehat{x}_{P_{1,Z^1_{1,1}}}}{dv} &= -\lambda(\min(\widehat{x}_{P_{1,Z^1_{1,1}}}, \widehat{x}_{P_{2,Z^1_{1,1}}}) - \lambda(\min(\widehat{x}_{P_{1,Z^1_{1,1}}}, \widehat{x}_{P_{2,Z^1_{1,1}}}) \\
\frac{d\widehat{x}_{P_{2,Z^1_{1,1}}}}{dv} &= -\lambda(\min(\widehat{x}_{P_{1,Z^1_{1,1}}}, \widehat{x}_{P_{2,Z^1_{1,1}}}) - \lambda(\min(\widehat{x}_{P_{1,Z^1_{1,1}}}, \widehat{x}_{P_{2,Z^1_{1,1}}}) \\
\frac{d\widehat{x}_{P_{3,Z^1_{1,1},Z^2_{1,1}}}}{dv} &= +\lambda(\min(\widehat{x}_{P_{1,Z^1_{1,1}}}, \widehat{x}_{P_{2,Z^1_{1,1}}})
\end{aligned}$$

[4] M. Beccuti, C. Fornari, G. Franceschinis, S. Halawani, O. Ba-Rukab, A. Ahmad, and G. Balbo. From symmetric nets to differential equations exploiting model symmetries. Computer Journal, 58(1):23-39, 2015.

Goal: efficient computation of measures

Exploit the regular structure of SSN models: use only a subset of ODEs, representing all the others, to derive the expected value of the interesting measures

Symbolic ODE

Exploit the calculus developed for structural properties computation of SSNs[5][6][7] to avoid unfolding and reduction of ODE system

Exploiting the calculus on arc expressions

The SODE for (unfolded) place *P[c]* contains a negative/positive term for each *transition instance* that causes a decrease/increase of c-colored tokens in *P*.

The SSN arc functions express in a *symbolic form* the state change that each transition instance causes on the colored tokens in its input and output places.

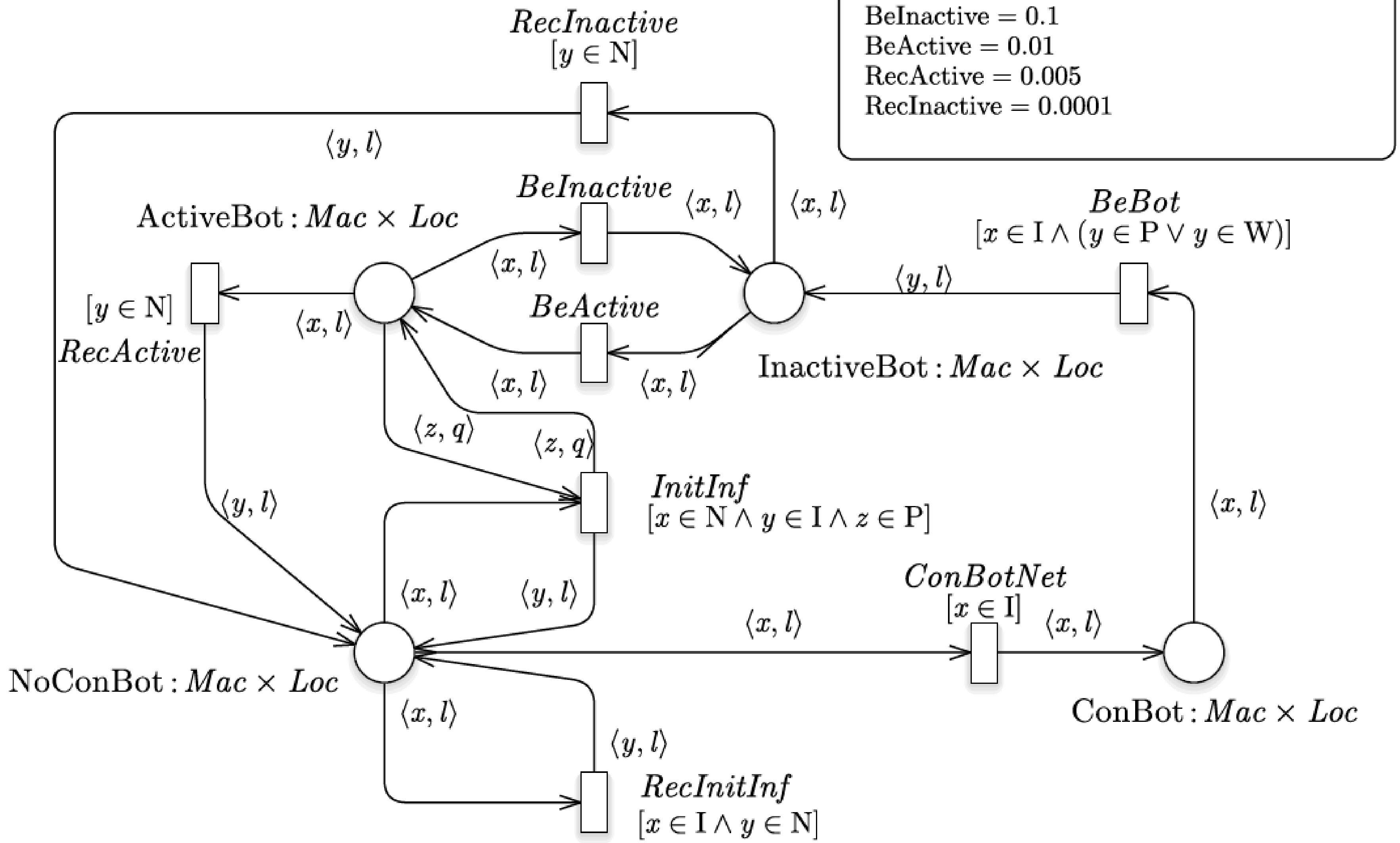
IDEA: This symbolic form can be used to find the (set of) transition instance(s) changing the number of c-colored tokens in *P*

class $Mac = n\{1..1\}$ **is** N $+\ i\{1..1\}$ **is** I $+\ q\{1..1\}$ **is** P $+\ w\{1..1\}$ **is** W
class $Loc = l\{1..10\}$

$$m_0 = \mathrm{NoConBot}\big(1000\,\langle \mathrm{N}, \boldsymbol{Loc}\rangle\big) + \mathrm{ConBot}\big(\langle \mathrm{I}, \boldsymbol{Loc}\rangle\big)$$

Rates

InitInf = same loc. 10; different loc. 2
BeBot = mach $\in$ W : 20, mach $\in$ P : 2
RecInitInf = 0.5
ConBotNet = 3
BeInactive = 0.1
BeActive = 0.01
RecActive = 0.005
RecInactive = 0.0001



Example inspired by [12]

Ordinary Differential Equations

Based on the UNFOLDED MODEL (SPN)

Two color classes: Mac = Machine states (N, I, P, W)

Loc = Local Area Network id

Size of the unfolded model:

- Places: $4 \cdot |Mac| \cdot |Loc| = 16 \cdot |Loc|$ (but $5 \cdot |Loc|$ isolated and empty)
- Transitions: $|Loc|$ (ConBot) + $2 \cdot |Loc|$ (BeBot) + $4 \cdot |Loc|$ (BeActive) + $4 \cdot |Loc|$ (BeInactive) + $4 \cdot |Loc|$ (RecActive) + $4 \cdot |Loc|$ (RecInactive) + $|Loc|$ (RecInitInf) + $|Loc|^2$ (InitInf) $= 20 \cdot |Loc| + |Loc|^2$

Size of the ODE system of the unfolded model:
$11|Loc|$ equations and $2 \cdot (20|Loc|+|Loc|^2)$ terms

ODE vs Symbolic ODE

|Loc|	Terms ODE	ODE/SODE	Mean Solution time(sec) ODE	SODE
1	42 (11 eq)	2.51 (1.57)	0.37	0.085
10	600 (110 eq.)	21.43 (15.7)	38.81	0.238
20	1600 (220 eq.)	57.14 (31.43)	572.18	0.292
50	7000 (550 eq)	250 (78.58)	>4h	0.248

Size of the ODE system of the unfolded model: $11 \cdot |Loc|$ equations and $2 \cdot (20 \cdot |Loc| + |Loc|^2)$ terms

Size of the SODE system for the same model: 7 SODE (+4 always = 0) with 28 terms: <u>independent of |Loc|</u>

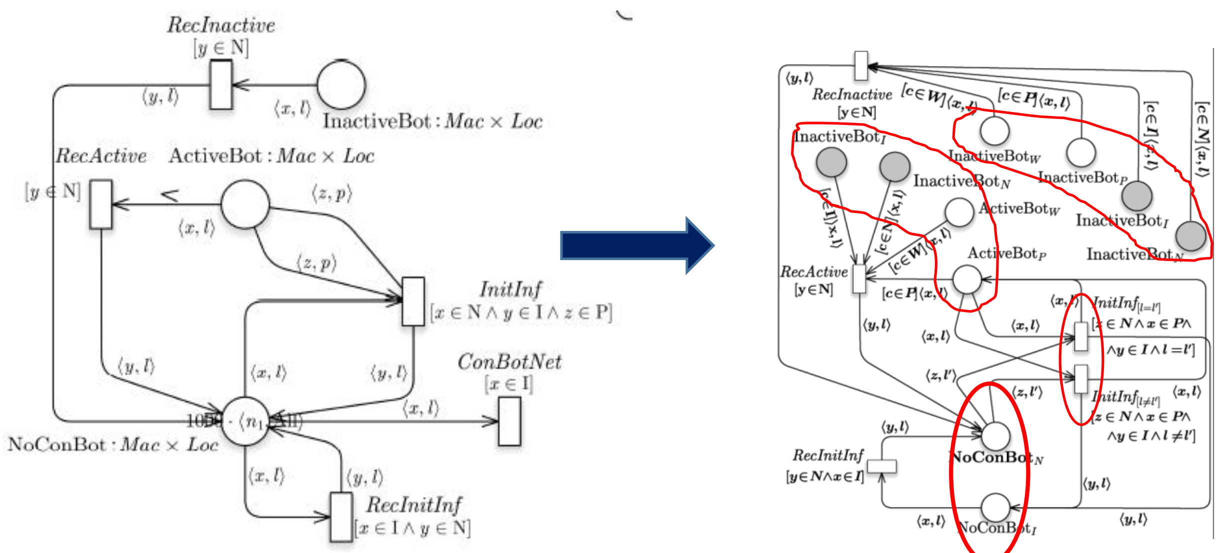
For large |Loc| the unfolded SPN grows very large and the approach becomes impractical

Steps to derive the system of SODE

- Partial unfolding of the places and transitions
- Derivation of «macro terms» in the symbolic ODE: functions Added-By and Removed-By, AB and RB
- Computation of the base rate and of the enabling degree of the terms: possible refinement of terms in subterms with homogeneous enabling degree
- Computation of the cardinality of the sets of transition instances represented by the refined terms (producing the same state changes with same rate)

Partial unfolding

- Partition of color classes into (static) subclasses
- Partition of place color domains based on color subclasses and comparison of color elements of same type ⇒ place partial unfolding
- Partition of transition instances based on their rate ⇒ transition partial unfolding



Sum of terms in the SODE

Added-By and Removed-By: transposing arc functions

$$\frac{dx[p,c]}{d\nu} = \sum_{\langle t,c'\rangle : p\in t^{\bullet}, c'\in \boxed{\mathcal{A}(t,p)(c)}} \varphi(x(\nu),t,c')\left(\overbrace{\boxed{\mathcal{A}(t,p)(c)[c']}}^{\text{Number of c-colored tokens added by } \langle t, c' \rangle \text{ in } p}\right) - \sum_{\langle t,c'\rangle : p\in {}^{\bullet}t, c'\in \boxed{\mathcal{R}(t,p)(c)}} \varphi(x(\nu),t,c')\left(\overbrace{\boxed{\mathcal{R}(t,p)(c)[c']}}^{\text{Number of c-colored tokens withdrawn by } \langle t, c' \rangle \text{ from } p}\right)$$

$$\left.\begin{aligned}
&\mathcal{A}(t,p) : cd(p) \to Bag[cd(t)];\ \mathcal{A}(t,p) = (O[t,p] - I[t,p])^{t} \\
&\mathcal{R}(t,p) : cd(p) \to Bag[cd(t)];\ \mathcal{R}(t,p) = (I[t,p] - O[t,p])^{t}
\end{aligned}\right\} \text{Expressed with the same syntax of arc functions}$$

place $(NoConBot_N, c, l)$;$g = c \in N$	
$\mathcal{A}(RecActive, .)$	$\langle S_{Mac}, c, l\rangle[g]$
$\mathcal{A}(RecInactive, .)$	$\langle S_{Mac}, c, l\rangle[g]$
$\mathcal{R}(InitInf_{[l=q]}, .)$	$\langle c, S_I, S_P, l, l\rangle[g]$
$\mathcal{R}(InitInf_{[l\neq q]}, .)$	$\langle c, S_I, S_P, l, S-l\rangle[g]$
$\mathcal{A}(RecInitInf, .)$	$\langle S_I, c_1, l\rangle[g]$

The terms may group several instances that add/withdraw the same number of tokens (e.g. $|Loc| - 1$ instances withdraw 1 token)

Enabling degree computation

- Composition of I/O arc fns with representative instance

$$\frac{dx[p,c]}{d\nu} = \sum_{\langle t,c'\rangle : p \in t^{\bullet}, c' \in \mathcal{A}(t,p)(c)} \boxed{\varphi(x(\nu),t,c')}(\mathcal{A}(t,p)(c)[c']) - \sum_{\langle t,c'\rangle : p \in {}^{\bullet}t, c' \in \mathcal{R}(t,p)(c)} \boxed{\varphi(x(\nu),t,c')}(\mathcal{R}(t,p)(c)[c']).$$

$$\varphi(x(\nu),t_j) = \omega(t_j) \min_{l:I[p_l,t_j]\neq 0} \frac{x_l(\nu)}{I[p_l,t_j]}$$

Intensity of t_j (where t_j corresponds to a transition instance $\langle t, c' \rangle$

The enabling degree of instance $\langle t, c' \rangle$ must be evaluated for the instances grouped in a term of $A(t,p)(c)$ or $R(t,p)(c)$ and *may not be uniform*

Partial unfolding ensures uniform rate $\omega(t_j)$ for all instances $\langle t, c' \rangle$ of any given t

Composition of Input arc fn. with a representative transition color instance provides the denominator; it may refine the set of instances into sub-sets (by rewriting the term as a sum of disjoint terms).

Cardinality of terms in $A(t,p)$ and $\mathrm{R}(t,p)$

$A(t,p)$ and $\mathrm{R}(t,p)$: sums of constant size terms (Property 2)

$$\frac{dx[p,c]}{d\nu} = \sum_{t:p\in t^{\bullet}, \boxed{F_i\ in\ \mathcal{A}(t,p)}} \lambda_i n_i \varphi(x(\nu),t) - \sum_{t:p\in {}^{\bullet}t, \boxed{F_j\ in\ \mathcal{R}(t,p)}} \lambda_j n_j \varphi(x(\nu),t)$$

$\boldsymbol{n_i = |F_i|, n_j = |F_j|}$

λ_i / λ_j multiplicity of $\boldsymbol{F_i/F_j}$ in $A(t,p)$ / $\mathrm{R}(t,p)$

For each term F_i representing several transition instances with uniform intensity φ, only one term is generated in the SODE and multiplied by $|F_i|$.

$$\frac{dx[NoConBot_N,c,l]}{d\nu} = |P|\omega_1 x[ActiveBot_P,c',l] + |W|\omega_2 x[ActiveBot_W,c',l] +$$
$$+ |P|\omega_3 x[InactiveBot_P,c',l] + |W|\omega_4 x[InactiveBot_W,c',l] + \omega_7 x[NoConBot_I,c',l] +$$
$$- |P||I|\omega_5 \min(x[NoConBot_I,c',l], x[ActiveBot_P,c'',l]) +$$
$$- |P||I|(|Loc|-1)\omega_6 \min(x[NoConBot_I,c',l], x[ActiveBot_P,c'',l]),$$

Tool support: www.di.unito.it/~depierro/SNex

- Library implementing algorithms for the computation of transpose, difference and a (restricted) composition operator. It is based on a set of rewriting rules.
- Command Line Interface (CLI)
  - CLI commands for – loading a SSN specification, «solving» an expression involving the above operators.
  - CLI commands to operate all the intermediate steps required to finally derive the Symbolic ODEs

Concluding remarks

- Implementation in SNexpression (to be completed)
- Second example in the Tech Report [3]
- The method works also for ordered classes

Related work and Future work

- Investigate the applicability to other formalism with similar symmetric structure [14](call for collaboration)
- Complete the implementation and extensive experimentation on large models
- Automatic precondition check of applicability exploiting the same symbolic calculus

[14] M. Tschaikowski and M. Tribastone. Exact fluid lumpability for Markovian process algebra. In M. Koutny and I. Ulidowski, editors, CONCUR 2012 pages 380-394. Springer Berlin Heidelberg, 2012.